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Two kinds of Novikov algebras and their realizations[☆]

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Abstract

In this paper, we construct two kinds of Novikov algebras, characterize some properties of them and give their realizations by triangle functions, respectively.

MSC: 17A30

1. Introduction

The Hamilton operator is an important operator of the calculus of variations. When Gel'fand and Dorfman [1–3] studied the following operator:

$$H_{ij} = \sum_k c_{ijk} u_k^{(1)} + d_{ijk} u_k^{(0)} \frac{\mathrm{d}}{\mathrm{d}x}, \quad c_{ijk} \in \mathbf{C}, d_{ijk} = c_{ijk} + c_{jik},$$

they gave the definition of Novikov algebras. Concretely, let c_{ijk} be the structural coefficients, a product of $L = L(e_0, e_1, \ldots)$ be $\circ$ such that

$$e_i \circ e_j = \sum c_{ijk} e_k.$$

Then the product is Hamilton operator if and only if $\circ$ satisfies

$$(a \circ b) \circ c = (a \circ c) \circ b$$
$$(a \circ b) \circ c + c \circ (a \circ b) = (c \circ b) \circ a + a \circ (c \circ b).$$

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In 1987, Zel'manov [4] began to study Novikov algebras and proved that the dimension of finite simple Novikov algebras over a field of characteristic zero is one. In algebras, what are paid attention to by mathematician are classifications and structures. We do not have the systematic theory for general Novikov algebras. In 1992, Osborn [5, 6] had finished the classification of infinite simple Novikov algebras with nilpotent elements over a field of characteristic zero and finite simple Novikov algebras with nilpotent elements over a field of characteristic $p > 0$. In 1995, Xu [7,10] developed his theory and got the classification of simple Novikov algebras over an algebraically closed field of characteristic zero. Bai and Meng [8,9] has serial work on low dimension Novikov algebras, such as the structure and classification.

Definition 1.1. Let $(\mathcal{A}, \circ)$ be an algebra over $\mathbf{F}$ such that:

$$(a, b, c) = (b, a, c), \tag{1.1}$$

$$(a \circ b) \circ c = (a \circ c) \circ b, \quad \forall a, b, c \in \mathcal{A}, \tag{1.2}$$

then $\mathcal{A}$ is called a Novikov algebra over $\mathbf{F}$.

Remark 1.1. (1) Condition (1.1) is usually written by

$$a \circ (b \circ c) - (a \circ b) \circ c = b \circ (a \circ c) - (b \circ a) \circ c. \tag{1.1$'$}$$

(2) An algebra $\mathcal{A}$ is called a left symmetric algebra if it only satisfies (1.1). It is clear that left symmetric algebras contain Novikov algebras.

Remark 1.2. (1) If $(\mathcal{A}, \circ)$ is a left symmetric algebra satisfying

$$[a, b] = a \circ b - b \circ a, \quad \forall a, b \in \mathcal{A}, \tag{1.3}$$

then $(\mathcal{A}, [\,,\,])$ is a Lie algebra. Usually, it is called an adjoining Lie algebra.

(2) Let $(\mathcal{A}, \cdot)$ be a commutative algebra, then $(\mathcal{A}, d_0, \circ)$ is a Novikov algebra if d_0 is a derivation of $\mathcal{A}$ with a bilinear operator $\circ$ such that

$$a \circ b = a \cdot d_0(b), \quad \forall a, b \in \mathcal{A}. \tag{1.4}$$

2. A kind of Novikov algebras and its realization

Lemma 2.1. *Let $\{b_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots\}$ be a basis of the linear space $\mathcal{A}$ over a field $\mathbf{F}$ of characteristic $p \neq 2$ satisfying*

$$\begin{cases} a_m a_n = -\dfrac{1}{2}(b_{m+n} - b_{m-n}), \\ b_m b_n = \dfrac{1}{2}(b_{m+n} + b_{m-n}), \\ a_m b_n = b_n a_m = \dfrac{1}{2}(a_{m+n} + a_{m-n}), \end{cases} \tag{2.1}$$

where $b_{-m} = b_m$, $a_{-m} = -a_m$. Then $\mathcal{A}$ is a commutative and associative algebra.

Proof. It is clear that $\mathcal{A}$ is a commutative algebra over $\mathbf{F}$.

$$\begin{aligned} (a_k, a_n, a_m) &= a_k(a_n a_m) - (a_k a_n) a_m \\ &= -\frac{1}{2} a_k (b_{m+n} - b_{n-m}) + \frac{1}{2}(b_{k+n} - b_{k-n}) a_m \\ &= \frac{1}{4}(-a_{k+m+n} - a_{k-m-n} + a_{k+n-m} + a_{k-n+m} + a_{m+k+n} + a_{m-k-n} - a_{m+k-n} - a_{m-k+n}) \\ &= 0. \end{aligned}$$

Similarly, $(b_k, b_n, b_m) = (a_k, a_n, b_m) = (a_k, b_n, a_m) = (b_k, a_n, a_m) = (b_k, b_n, a_m) = (b_k, a_n, b_m) = (a_k, b_n, b_m) = 0$. Then $(a, b, c) = 0, \forall a, b, c \in \mathcal{A}$. The result follows. $\square$

Corollary 2.2. *b_0 of Lemma 2.1 is a unity of $\mathcal{A}$.*

Lemma 2.3. *Let $\mathcal{A}$ be a commutative and associative algebra satisfying Lemma 2.1. Then the following statements hold:*

(1) *If D_0 is a linear transformation of $\mathcal{A}$ such that*

$$\begin{cases} D_0(a_n) = nb_n, & n = 1, 2, 3, \dots, \\ D_0(b_n) = -na_n, & n = 0, 1, 2, \dots; \end{cases} \tag{2.2}$$

then D_0 is a derivation of $\mathcal{A}$.

(2) *If aD_0 is a linear transformation of $\mathcal{A}$ such that*

$$(aD_0)(b) = aD_0(b), \quad \forall a, b \in \mathcal{A}, \tag{2.3}$$

then aD_0 is a derivation of $\mathcal{A}$.

(3) *$\mathcal{D}_1 = \{aD_0 | a \in \mathcal{A}\}$ is a subalgebra of Lie algebra* Der $\mathcal{A}$.

Proof. (1) We have

$$\begin{aligned} D_0(a_n a_m) &= D_0\left(-\frac{1}{2}(b_{n+m} - b_{n-m})\right) \\ &= \frac{1}{2}((m+n)a_{n+m} - (n-m)a_{n-m}); \\ D_0(a_n)a_m + a_n D_0(a_m) &= nb_n a_m + ma_n b_m \\ &= \frac{n}{2}(a_{n+m} - a_{n-m}) + \frac{m}{2}(a_{n+m} - a_{m-n}) \\ &= \frac{1}{2}((m+n)a_{m+n} - (n-m)a_{n-m}). \end{aligned}$$

So D_0 is a derivation of $\mathcal{A}$.

(2) For $\forall a, b, c \in \mathcal{A}$, we have $(aD_0)(bc) = aD_0(bc) = aD_0(b)c + abD_0(c) = (aD_0)(b)c + b(aD_0)(c)$, so aD_0 is a derivation of $\mathcal{A}$.

(3) For $\forall a, b, c \in \mathcal{A}$, we have

$$\begin{aligned} [aD_0, bD_0](c) &= (aD_0)(bD_0)(c) - (bD_0)(aD_0)(c) \\ &= aD_0(b)D_0(c) - bD_0(a)D_0(c) \\ &= (aD_0(b) - bD_0(a))D_0(c), \end{aligned}$$

so, $[aD_0, bD_0] = (aD_0(b) - bD_0(a))D_0 \in \mathcal{D}_1$. Hence, (3) holds. □

Theorem 2.4. *Let $\mathcal{A}$ be a commutative and associative algebra satisfying Lemma 2.1, and let a be an element of $\mathcal{A}$. If D_0 satisfies Lemma 2.3 and $\circ$ satisfies*

$$b \circ c = baD_0(c), \quad \forall b, c \in \mathcal{A}, \tag{2.5}$$

then the following statements hold:

(1) *$(\mathcal{A}, aD_0, \circ)$ is a Novikov algebra.*

(2) *$(\mathcal{A}, aD_0, [,])$ is an adjoining Lie algebra of $(\mathcal{A}, aD_0, \circ)$ and $[,]$ such that*

$$[b, c] = a(bD_0(c) - cD_0(b)), \quad \forall b, c \in \mathcal{A}. \tag{2.6}$$

Proof. (1) By Lemma 2.3, aD_0 is a derivation of the commutative algebra $\mathcal{A}$. So $(\mathcal{A}, aD_0, \circ)$ is a Novikov algebra by Remark 1.2(2).

(2) $(\mathcal{A}, aD_0, [,])$ is an adjoining Lie algebra of $(\mathcal{A}, aD_0, \circ)$ by Remark 1.2(1). For $\forall b, c \in \mathcal{A}, \exists a \in \mathcal{A}$, we have $[b, c] = b \circ c - c \circ b = baD_0(c) - caD_0(b) = a(bD_0(c) - cD_0(b))$ since $\mathcal{A}$ is commutative. Hence we obtain the desired result. □

Let b_0 be a unity of $\mathcal{A}$. If we set $a = b_0$ in Theorem 2.4, then $a_n \circ a_m = a_n b_0 D_0(a_m) = a_n(mb_m) = \frac{m}{2}(a_{m+n} - a_{n-m})$. Similarly, we obtain Corollary 2.5.

Corollary 2.5. *Let $\mathcal{A}$ be a commutative and associative algebra satisfying Lemma 2.1. Then the following statements hold:*

$$\begin{cases} a_n \circ a_m = \frac{m}{2}(a_{n+m} + a_{n-m}), \\ b_n \circ b_m = -\frac{m}{2}(a_{n+m} + a_{m-n}), \\ a_n \circ b_m = \frac{m}{2}(b_{n+m} - b_{n-m}), \\ b_n \circ a_m = \frac{m}{2}(b_{n+m} + b_{n-m}). \end{cases} \tag{2.7}$$

$$\begin{cases} [a_n, a_m] = \frac{1}{2}(m-n)a_{n+m} + \frac{1}{2}(m+n)a_{n-m}, \\ [b_n, b_m] = \frac{1}{2}(n-m)a_{n+m} + \frac{1}{2}(m+n)a_{n-m}, \\ [a_n, b_m] = \frac{1}{2}(m-n)b_{n+m} + \frac{1}{2}(n-m)b_{n-m}, \\ [b_n, a_m] = \frac{1}{2}(m-n)b_{n+m} + \frac{1}{2}(m-n)b_{n-m}. \end{cases} \tag{2.8}$$

We will construct Novikov algebras over the linear space which is generated by triangle functions. The field $\mathbf{F}$ *is assumed* $\mathbf{R}$ *or* $\mathbf{C}$ *from Lemma 2.6 to Theorem 2.9.*

First, let $\mathcal{T}$ be a linear space generated by $\{\sin mx, \cos nx | m, n \in \mathbf{Z}\}$ over $\mathbf{R}$. $\langle , \rangle$ satisfying

$$\langle f(x), g(x)\rangle = \int_0^{2\pi} f(x)g(x)\mathrm{d}x, \quad \forall f(x), g(x) \in \mathcal{T}. \tag{2.9}$$

Clearly, $\langle , \rangle$ is symmetric, bilinear and positive definite, so $\mathcal{T}$ can be seen to be a Euclidean space over $\mathbf{F}$.

Lemma 2.6. *$\mathcal{T}$ satisfying the above product is a commutative associative algebra.*

Proof. Since the above product is commutative and associative, we only need that $\mathcal{T}$ be closed for the product. In fact,

$$\begin{cases} \sin mx \sin nx = -\frac{1}{2}(\cos(m+n)x - \cos(m-n)x) \\ \cos mx \cos nx = \frac{1}{2}(\cos(m+n)x + \cos(m-n)x) \\ \sin mx \cos nx = \frac{1}{2}(\sin(m+n)x + \sin(m-n)x). \end{cases} \tag{2.10}$$

So $\mathcal{T}$ is a commutative and associative algebra. □

Lemma 2.7. $\{1, \sin nx, \cos mx | n, m \in \mathbf{N}\}$ *is an orthogonal system and a basis of $\mathcal{T}$ over* $\mathbf{F}$.

Proof. For $m \in \mathbf{N}$, we have

$$\int_0^{2\pi} \sin mx\mathrm{d}x = \int_0^{2\pi} \cos mx\mathrm{d}x = 0.$$

By (2.9) and (2.10), we obtain

$$\langle \sin mx, \sin nx\rangle = \langle \cos mx, \cos nx\rangle = \delta_{mn}\pi,$$
$$\langle \sin mx, \cos nx\rangle = 0.$$

So $\{1, \sin nx, \cos mx | n, m \in \mathbf{N}\}$ is an orthogonal system of $\mathcal{T}$. Moreover, $\mathcal{T}$ is generated by $\{\sin mx, \cos nx | m, n \in \mathbf{Z}\}$. $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, hence the result follows. □

Theorem 2.8. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be commutative and associative algebras over* $\mathbf{F}$. *If $\varphi: \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is an isomorphism and $D_1 \in \mathrm{Der}\,\mathcal{A}_1$, then the following statements hold:*

(1) $D_2 := \varphi D_1 \varphi^{-1} \in \operatorname{Der} \mathcal{A}_2$.
(2) *φ: $(\mathcal{A}_1, D_1, \circ) \longrightarrow (\mathcal{A}_2, D_2, \circ)$ is also an isomorphism of Novikov algebras.*

Proof. (1) For $\forall a, b \in \mathcal{A}_1$, we have

$$\begin{aligned}(\varphi D_1 \varphi^{-1})(\varphi(a)\varphi(b)) &= (\varphi D_1 \varphi^{-1})(\varphi(ab)) \\ &= \varphi D_1(ab) = \varphi(D_1(a)b + aD_1(b)) = \varphi(D_1(a))\varphi(b) + \varphi(a)\varphi(D_1(b)) \\ &= (\varphi D_1 \varphi^{-1})(\varphi(a))\varphi(b) + \varphi(a)(\varphi D_1 \varphi^{-1})(\varphi(b)).\end{aligned}$$

So (1) holds.

(2) For $\forall a, b \in \mathcal{A}_1$, we have

$$\begin{aligned}\varphi(a \circ b) &= \varphi(aD_1(b)) = \varphi(a)\varphi(D_1(b)) \\ &= \varphi(a)(\varphi D_1 \varphi^{-1})(\varphi(b)) = \varphi(a)D_2(\varphi(b)) \\ &= \varphi(a) \circ \varphi(b).\end{aligned}$$

So (2) holds. □

Theorem 2.9. *Let $\mathcal{A}$ be a commutative and associative algebra over $\mathbf{R}$ satisfying Lemma 2.1, D_0 be its derivation satisfying (2.2) and $\mathcal{T}$ be a commutative and associative algebra over $\mathbf{R}$ satisfying Lemmas 2.6 and 2.7. If $\varphi : \mathcal{A} \longrightarrow \mathcal{T}$ satisfies*

$$\varphi(b_m) = \cos mx, \quad m = 0, 1, 2, \ldots, \qquad \varphi(a_n) = \sin nx, \quad n = 1, 2, \ldots, \tag{2.11}$$

then the following statements hold:

(1) *φ is an isomorphism of commutative and associative algebras.*
(2) $\varphi D_0 \varphi^{-1} = \frac{\mathrm{d}}{\mathrm{d}x}$.
(3) *$\varphi : (\mathcal{A}, aD_0, \circ) \longrightarrow (\mathcal{T}, \varphi(a)\frac{\mathrm{d}}{\mathrm{d}x}, \circ)$ is an isomorphism of Novikov algebras.*

Proof. (1) It is clear by Lemma 2.7, (2.1) and (2.10).

(2) By (2.2) and (2.11), we have

$$\begin{aligned}\varphi D_0 \varphi^{-1}(\sin nx) &= \varphi D_0(a_n) \\ &= \varphi(nb_n) = n\cos nx \\ &= \frac{\mathrm{d}\sin nx}{\mathrm{d}x}, \\ \varphi D_0 \varphi^{-1}(\cos nx) &= \varphi D_0(b_n) \\ &= \varphi(-na_n) = -n\sin nx \\ &= \frac{\mathrm{d}\cos nx}{\mathrm{d}x}.\end{aligned}$$

So (2) holds.

(3) It is clear that $\varphi(aD_0)\varphi^{-1} = \varphi(a)\mathrm{d}/\mathrm{d}x$. By (2.11) and (2.2), we have

$$\begin{aligned}\varphi(aD_0)\varphi^{-1}(\sin nx) &= \varphi(aD_0)(a_n) \\ &= \varphi(aD_0(a_n)) = \varphi(anb_n) \\ &= \varphi(a)\varphi(nb_n) = \varphi(a)n\cos nx \\ &= \varphi(a)\mathrm{d}(\sin nx)/\mathrm{d}x.\end{aligned}$$

Similarly, we have $\varphi(aD_0)\varphi^{-1}(\cos nx) = \varphi(a)\mathrm{d}(\cos nx)/\mathrm{d}x$. So $\varphi(aD_0)\varphi^{-1} = \varphi(a)\mathrm{d}/\mathrm{d}x$.

By Theorems 2.4 and 2.8 and Remark 1.2(2), we have

$$\begin{aligned}\varphi(b \circ c) &= \varphi(baD_0(c)) \\ &= \varphi(b)\varphi(aD_0(c)) \\ &= \varphi(b)[\varphi(aD_0)\varphi^{-1}(\varphi(c))] \\ &= \varphi(b)\varphi(a)\mathrm{d}/\mathrm{d}x(\varphi(c)) \\ &= \varphi(b) \circ \varphi(c), \quad \forall b, c \in \mathcal{A}.\end{aligned}$$

So $\varphi : (\mathcal{A}_0, aD_0, \circ) \longrightarrow (\mathcal{T}, \varphi(a)\frac{\mathrm{d}}{\mathrm{d}x}, \circ)$ is an isomorphism of Novikov algebras. $\square$

3. Another kind of Novikov algebras and its realization

Lemma 3.1. *Let $\{c_0, c_{\pm 1}, c_{\pm 2}, \ldots, c_{\pm n}, \ldots\}$ be a basis of $\mathcal{B}$ over* **F** *and the product of $\mathcal{B}$ be defined by*

$$c_m c_n = c_{m+n}. \tag{3.1}$$

Then $\mathcal{B}$ is a commutative and associative algebra, $c_0 = 1$ and c_0 is an unity of $\mathcal{B}$.

Proof. It is clear by means of a routine computation. $\square$

Lemma 3.2. *Let $\mathcal{B}$ be a commutative and associative algebra satisfying Lemma 3.1. Then the following statements hold.*

(1) *If D_0 is a linear transformation of $\mathcal{B}$ such that*

$$D_0(c_m) = mc_{m-1}, \tag{3.2}$$

then D_0 is a derivation of $\mathcal{B}$.

(2) *For $c \in \mathcal{B}$, $\mathrm{Der}(\mathcal{B}) = \{cD_0 | c \in \mathcal{B}\}$.*

Proof. (1) For any elements c_m, c_n of a basis of $\mathcal{B}$, we have

$$\begin{aligned} D_0(c_m c_n) &= D_0(c_{m+n}) = (m+n)c_{m+n-1} \\ &= mc_{m-1}c_n + nc_m c_{n-1} \\ &= D_0(c_m)c_n + c_m D_0(c_n), \end{aligned}$$

so (1) holds.

(2) By the definition of $\mathcal{B}$, we can see that $\mathcal{B}$ is generated by c_1 and c_{-1}, and all derivations of $\mathcal{B}$ are determined by the effect on c_1 and c_{-1}.

$\forall D \in \mathrm{Der}(\mathcal{B})$, there is $c \in \mathcal{B}$ such that $D(c_1) = c$. Clearly, cD_0 is a derivation of $\mathcal{B}$ and $c_1 c_{-1} = c_0 = 1$, then $D(c_1)c_{-1} + c_1 D(c_{-1}) = D(c_1 c_{-1}) = D(1) = 0$, so $D(c_{-1}) = -c_{-2}D(c_1)$.

By (3.2), we have

$$(D - cD_0)(c_1) = D(c_1) - cD_0(c_1) = c - c = 0$$

and

$$(D - cD_0)(c_{-1}) = D(c_{-1}) - cD_0(c_{-1}) = -c_{-2}D(c_1) + cc_{-2} = 0,$$

then $D = cD_0$ since all derivations of $\mathcal{B}$ are determined by the effect on c_1 and c_{-1}. Hence we obtain the desired result. $\square$

Now we will realize the Novikov algebra above by a concrete linear space. Let $\mathcal{L}$ be a linear space over **R** generated by $\{\mathrm{e}^{\pm n\sqrt{-1}x}, | n \in \mathbf{N}\}$.

Lemma 3.3. *$\mathcal{L}$ is a commutative and associative algebra over* **R** *and $\{1, \mathrm{e}^{\pm n\sqrt{-1}x}, | n \in \mathbf{N}_0\}$ is a basis of $\mathcal{L}$ over* **R**.

Proof. Since $\mathrm{e}^{\pm m\sqrt{-1}x}\mathrm{e}^{\pm n\sqrt{-1}x} = \mathrm{e}^{\pm(m+n)\sqrt{-1}x}$, $n \in \mathbf{N}$, the product of $\mathcal{L}$ is closed. Hence it is clear that $\mathcal{L}$ is a commutative and associative algebra.

For $\forall n \in \mathbf{N}$, if there is $a_{\pm i} \in \mathbf{R}$ such that

$$\begin{aligned} &a_{-n}\mathrm{e}^{-n\sqrt{-1}x} + a_{-(n-1)}\mathrm{e}^{-(n-1)\sqrt{-1}x} + \cdots + a_{-1}\mathrm{e}^{-\sqrt{-1}x} + a_0 + a_1\mathrm{e}^{\sqrt{-1}x} + a_2\mathrm{e}^{2\sqrt{-1}x} + \cdots \\ &\quad + a_n\mathrm{e}^{n\sqrt{-1}x} = 0, \end{aligned} \tag{3.3}$$

then put $x = \frac{\pi}{2}$, we have $a_0 = 0$.

Let $x = 0$,

$$a_{-n} + a_{-(n-1)} + \cdots + a_{-1} + a_0 + a_1 + a_2 + \cdots + a_n = 0.$$

We take the derivative for (3.3) such that its derivative order is $2k$ ($k \in N$), and put $x = 0$. Then we have

$$n^{2k}(a_n + a_{-n}) + (n-1)^{2k}(a_{-(n-1)} + a_{n-1}) + \cdots + (a_1 + a_{-1}) = 0.$$

Let $k = 0, 1, \ldots, n-1$, then we have the following equation groups:

$$\begin{cases} (a_1 + a_{-1}) + (a_2 + a_{-2}) + \cdots + n(a_n + a_{-n}) = 0, \\ (a_1 + a_{-1}) + 2^2(a_2 + a_{-2}) + \cdots + n^2(a_n + a_{-n}) = 0, \\ (a_1 + a_{-1}) + 2^4(a_2 + a_{-2}) + \cdots + n^4(a_n + a_{-n}) = 0, \\ \ldots, \\ (a_1 + a_{-1}) + 2^{2(n-1)}(a_2 + a_{-2}) + \cdots + n^{2(n-1)}(a_n + a_{-n}) = 0. \end{cases} \tag{3.4}$$

If $a_i + a_{-i}$ are seen to be unknown, then the coefficient matrix of (3.4) is the Vandermonde matrix whose determinant is not 0, so $a_i + a_{-i} = 0$, and $a_{-i} = -a_i$.

We take the derivative for (3.3) such that its derivative order is $2k-1$ ($k \in N$), and put $x = 0$. We have the following equation groups:

$$n^{2k-1}(a_n - a_{-n}) + (n-1)^{2k-1}(a_{(n-1)} - a_{-(n-1)}) + \cdots + (a_1 - a_{-1}) = 0.$$

Let $k = 0, 1, \ldots, n$, then we have the following equation groups by $a_{-i} = -a_i$:

$$\begin{cases} a_1 + 2a_2 + \cdots + na_n = 0, \\ a_1 + 2^3 a_2 + \cdots + n^3 a_n = 0, \\ a_1 + 2^5 a_2 + \cdots + n^5 a_n = 0, \\ \ldots, \\ a_1 + 2^{2n-1} a_2 + \cdots + n^{2n-1} a_n = 0, \end{cases} \tag{3.5}$$

the coefficient matrix of (3.5) is the Vandermonde matrix whose determinant is not 0, so $a_i = 0$ and $a_{\pm i} = 0$. Let $n \longrightarrow \infty$, then $\{1, \mathrm{e}^{\pm n\sqrt{-1}x}, |n \in \mathbf{N}_0\}$ are linear independent over $\mathbf{R}$. Hence the result follows. □

Theorem 3.4. *Let $\mathcal{B}$ be a commutative and associative algebra over $\mathbf{R}$ satisfying Lemma 3.1, D_0 be a derivation of $\mathcal{B}$ satisfying (3.2); $\mathcal{L}$ be a commutative and associative algebra over $\mathbf{R}$ satisfying Lemma 3.3. If f is a linear transformation of $\mathcal{L}\mathrm{e}^{n\sqrt{-1}x} \to \mathrm{e}^{(n-1)\sqrt{-1}x}$ and $\varphi : \mathcal{B} \longrightarrow \mathcal{L}$ such that $\varphi(c_n) = \mathrm{e}^{n\sqrt{-1}x}$ $n = 0, \pm 1, \pm 2, \ldots$, then the following statements hold:*

(1) *φ is an isomorphism.*
(2) $\varphi D_0 \varphi^{-1} = f \circ \frac{\mathrm{d}}{\sqrt{-1}\mathrm{d}x}$.
(3) *φ: $(\mathcal{B}, cD_0, \circ) \longrightarrow (\mathcal{L}, \varphi(c) f \circ \frac{\mathrm{d}}{\sqrt{-1}\mathrm{d}x}, \circ)$ is an isomorphism of Novikov algebras.*

Proof. (1) It is clear by Lemmas 3.1 and 3.3.
(2)

$$\begin{aligned} \varphi D_0 \varphi^{-1}(\mathrm{e}^{n\sqrt{-1}x}) &= \varphi D_0(c_n) \\ &= \varphi(nc_{n-1}) = n(\mathrm{e}^{(n-1)\sqrt{-1}x}) \\ &= f \circ \frac{\mathrm{d}\mathrm{e}^{n\sqrt{-1}x}}{\sqrt{-1}\mathrm{d}x}, \end{aligned}$$

so (2) holds.
(3) It is clear that $\varphi(cD_0)\varphi^{-1} = \varphi(c) f \frac{\mathrm{d}}{\sqrt{-1}\mathrm{d}x}$ by (2). So (3) holds by Theorem 2.8. □

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References

[1] I.M. Gel'fand, L.A. Diki, Asymptotic behavior of the resolvent of Sturm–Liouville equations and the Lie algebra of the Korteweg–de Vries equations, Russian Math. Surveys 30 (1975) 77–113.
[2] I.M. Gel'fand, L.A. Diki, A Lie algebra structure in a formal variational calculation, Funct. Anal. Appl. 10 (1976) 16–22.
[3] I.M. Gel'fand, I.Ya. Dorfman, Hamiltonian operators and algebraic structures related to them, Funct. Anal. Appl. 13 (1979) 248–262.
[4] E.I. Zel'manov, On a class of local translation invariant Lie algebras, Sov. Math. Dokl. 35 (1) (1987) 216–218.
[5] J.M. Osborn, Infinite dimensional Novikov algebras of characteristic 0, J. Algebra 167 (1994) 146–167.
[6] J.M. Osborn, Simple Novikov algebras with an idempotent, Comm. Algebra 20 (1992) 2729–2753.
[7] X. Xu, On simple Novikov algebra and their irreducible modules, J. Algebra 185 (1996) 905–934.
[8] C.M. Bai, D.J Meng, The classification of Novikov algebras in low dimensions, J. Phys. A 34 (2001) 1581–1594.
[9] C.M. Bai, D.J. Meng, The realizations of non-transitive Novikov algebras, J. Phys. A 34 (2001) 6435.
[10] Xiaoping Xu, Novikov–Poisson algebras, J. Algebra 190 (1997) 253–279.